Probabilistic Methods in Combinatorics

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Assignment 3

To solve for the Example class on 11th March. Submit the solution of Problem 2 by Sunday 9th March if you wish feedback on it.

The solution of each problem should be no longer than one page!

Problem 1. Let H be a graph and let $n > |V(H)|$ be an integer. Suppose that there is a graph on n vertices and m edges that does not contain a copy of H, and let $k > \frac{n^2 \log n}{m}$. Show that the edges of K_n can be coloured with k colours such that there is no monochromatic copy of H.

Problem 2. Show that there is a positive constant $c > 0$ such that for any positive integer n there exists a graph $G = (V, E)$ such that

- $|V| = n$,
- $|E| \geq cn^{8/7}$,
- G does not contain C_8 as a subgraph.

Problem 3. A collection $\mathcal{F}$ of subsets of $[n]$ is called *k-independent* if for every k distinct sets $F_1, \dots, F_k \in \mathcal{F}$, all of the 2^k intersections $\bigcap_{i=1}^k G_i$ are non-empty, where each G_i is either F_i or its complement $[n] \setminus F_i$. Prove that for $k \geq 6$ there is a k-independent family of subsets of $[n]$ of size at least $\left\lfloor e^{n/(k2^k)} \right\rfloor$ (exponentially large!).

Problem 4. Let $G = (V, E)$ be a graph on n vertices, with minimum degree $\delta > 1$. We say that a set $U \subseteq V$ is dominating if every vertex $v \in V \setminus U$ has at least one neighbour in U. Show that G has a dominating set of size at most $\frac{\log(\delta+1)+1}{\delta+1} n$.